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Rank-one operators in reflexive one-sided $\mathcal{A}$-submodules

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Abstract. In this paper, we first characterize reflexive one-sided $\mathscr{A}$-submodules $\mathscr{U}$ of a unital operator algebra $\mathscr{A}$ in $\mathscr{B}(\mathscr{H})$ completely. Furthermore we investigate the invariant subspace lattice $\operatorname{Lat}\mathscr{R}$ and the reflexive hull $\operatorname{Ref}\mathscr{R}$, where $\mathscr{R}$ is the submodule generated by rank-one operators in $\mathscr{U}$; in particular, if $\mathscr{L}$ is a subspace lattice, we obtain when the rank-one algebra $\mathscr{R}$ of $\operatorname{Alg}\mathscr{L}$ is big enough to determined $\operatorname{Alg}\mathscr{L}$ in the following senses: $\operatorname{Alg}\mathscr{L} = \operatorname{Alg}\operatorname{Lat}\mathscr{R}$ and $\operatorname{Alg}\mathscr{L} = \operatorname{Ref}\mathscr{R}$.

Keywords. Reflexive one-sided $\mathscr{A}$-submodule; rank-one operator.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$ and $\mathscr{P}$ the complete lattice of all orthogonal projections in $\mathscr{B}(\mathscr{H})$. Suppose that $\mathscr{A}$ is a unital operator algebra in $\mathscr{B}(\mathscr{H})$ and ϕ is an order homomorphism of $\operatorname{Lat}\mathscr{A}$ into itself (i.e. $E \le F$ implies $\phi(E) \le \phi(F)$), where $\operatorname{Lat}\mathscr{A}$ is the complete lattice of all invariant projections for $\mathscr{A}$. Then the set $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \phi(E)$ for all $E \in \operatorname{Lat}\mathscr{A}\}$ is clearly a weakly closed two-sided $\mathscr{A}$-submodule of $\mathscr{B}(\mathscr{H})$.

It became apparent that many interesting classes of non-self adjoint operator algebras arise as just such a module. Erdos and Power in [3] proved that any weakly closed $\mathscr{A}$-submodule of $\mathscr{B}(\mathscr{H})$ for a nest algebra $\mathscr{A}$ is of the above form. In [4], Han Deguang proved that this is also true for any reflexive algebra $\mathscr{A}$, which is σ-weakly generated by rank-one operators in itself. The purpose of this paper is to show that any reflexive right $\mathscr{A}$-submodule and $*$-reflexive left $\mathscr{A}$-submodule of a unital operator algebra $\mathscr{A}$ are determined by order homomorphisms from $\operatorname{Lat}\mathscr{A}$ into $\mathscr{P}$. As a corollary, we obtain the complete characterization of all σ-weakly closed one-sided $\mathscr{A}$-submodules, where $\mathscr{A}$ is σ-weakly generated by rank-one operators in itself or, in particular, $\mathscr{A}$ is a nest algebra.

In [2], Erdos showed that if $\operatorname{Lat}\mathscr{A}$ is a nest then the set of finite sums of rank-one operators in $\mathscr{A}$ is σ-weakly dense in $\mathscr{A}$. In [9], Longstaff asked whether the same conclusion holds for the more general case of completely distributive lattices, and showed that, in the opposite direction, complete distributivity is a necessary condition for this. Subsequently, Lambrou [6] showed that complete distributivity of the invariant subspace lattices implies a condition somewhat weaker than the strong density. Laurie and Longstaff [7] proved that the answer is affirmative if additional requirement of commutativity is imposed on the invariant subspace lattice. In §3, we will consider when the rank-one subalgebra $\mathscr{R}$ of $\operatorname{Alg}\mathscr{L}$ determines $\operatorname{Alg}\mathscr{L}$ in senses other than the σ-weak density.

Which subspace lattices $\mathscr{L}$ are determined by the rank-one subalgebra $\mathscr{R}$ of $\operatorname{Alg}\mathscr{L}$ in the sense that $\mathscr{L} = \operatorname{Lat}\mathscr{R}$? This question was answered by Longstaff in ([8], Proposition 3.2). A sufficient but not necessary condition ([8], Corollary 3.2.1) was given and it is shown in [8] that this condition is strictly weaker than complete distributivity. In §3, we investigate the invariant subspace lattice of the rank-one submodule of $\mathscr{U}$. As an application, we derive the sufficient and necessary condition obtained by Longstaff in [8] in order that $\mathscr{L} = \operatorname{Lat}\mathscr{R}$. As another application, we also obtain an equivalent condition for which $\operatorname{Alg}\mathscr{L} = \operatorname{Alg}\operatorname{Lat}\mathscr{R}$.

In §3, we also study when the rank-one submodule $\mathscr{R}$ of a reflexive one-sided $\mathscr{A}$-submodule $\mathscr{U}$ is big enough to determine $\mathscr{U}$ in the sense that $\operatorname{Ref}\mathscr{R} = \mathscr{U}$, where $\operatorname{Ref}\mathscr{R} = \{T \in \mathscr{B}(\mathscr{H}) : Tx \in [\mathscr{R}x] \text{ for all } x \in \mathscr{H}\}$ is the reflexive hull of $\mathscr{R}$. An equivalent condition for $\operatorname{Ref}\mathscr{R} = \mathscr{U}$ is given by means of order homomorphisms from $\operatorname{Lat}\mathscr{A}$ into $\mathscr{P}$.

The terminology and notation of this paper concerning reflexive subspaces may be found in [5]. In what follows, we always assume that $\mathscr{A}$ is a unital operator algebra in $\mathscr{B}(\mathscr{H})$. Set

$$\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P}) = \{\phi : \phi \text{ is an order homomorphism from } \operatorname{Lat}\mathscr{A} \text{ into } \mathscr{P}\}.$$

Given ϕ in $\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$, a right $\mathscr{A}$-submodule is associated which is given by

$$\mathscr{U}_\phi^r = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \phi(E), \forall E \in \operatorname{Lat}\mathscr{A}\};$$

and a left $\mathscr{A}$-submodule which is given by

$$\mathscr{U}_\phi^l = \{T \in \mathscr{B}(\mathscr{H}) : T\phi(E) \subseteq E, \forall E \in \operatorname{Lat}\mathscr{A}\}.$$

Clearly they are weakly closed. We say that $\mathscr{U}_\phi^r$ (and $\mathscr{U}_\phi^l$) are the right(left) $\mathscr{A}$-submodule determined by ϕ respectively. To each ϕ in $\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ there is naturally associated $\phi_\sim$ in $\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ given by

$$\phi_\sim(E) = \vee\{F \in \operatorname{Lat}\mathscr{A} : \phi(F) \not\geq E\}, \quad \forall E \in \operatorname{Lat}\mathscr{A}$$

(with the convention that $\phi_\sim(0) = 0$). Observe that $\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ has a natural partial ordering given by $\phi \leq \psi$ if and only if $\phi(E) \leq \psi(E)$ for any $E \in \operatorname{Lat}\mathscr{A}$. It follows that $\phi \leq \psi$ implies $\phi_\sim \geq \psi_\sim$.

2. Basic properties of one-sided $\mathcal{A}$-submodules

A subspace $\mathscr{S}$ of $\mathscr{B}(\mathscr{H})$ is said to be $*$-reflexive, if $\mathscr{S}^*$ is reflexive.

Theorem 2.1. *Suppose that $\mathscr{A}$ is a unital operator algebra in $\mathscr{B}(\mathscr{H})$ and $\mathscr{U}$ is a subspace of $\mathscr{B}(\mathscr{H})$. Then*

(1) *$\mathscr{U}$ is a reflexive right $\mathscr{A}$-submodule if and only if there exists $\phi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ such that $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \phi(E), \forall E \in \operatorname{Lat}\mathscr{A}\}$;*
(2) *$\mathscr{U}$ is a $*$-reflexive left $\mathscr{A}$-submodule if and only if there exists $\psi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ such that $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : T\psi(E) \subseteq E, \forall E \in \operatorname{Lat}\mathscr{A}\}$.*

Proof. (1) *Sufficiency.* Clearly $\mathscr{U}$ is a right $\mathscr{A}$-submodule, so we only need to prove that $\mathscr{U}$ is reflexive. Suppose that $T \in \mathscr{B}(\mathscr{H})$ and $Tx \in [\mathscr{U}x]$ for any $x \in \mathscr{H}$. Thus for any $E \in \operatorname{Lat}\mathscr{A}$,

$$TE \subseteq [\mathscr{U}E] = [\phi(E)\mathscr{U}E] = \phi(E)[\mathscr{U}E] \subseteq \phi(E).$$

So $T \in \mathscr{U}$ and it shows that $\mathscr{U}$ is reflexive.

Necessity. For any $E \in \operatorname{Lat}\mathscr{A}$, let $\phi(E) = [\mathscr{U}E]$. Clearly ϕ is an order homomorphism in $\operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$. Set

$$\mathscr{U}_\phi^r = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \phi(E), \forall E \in \operatorname{Lat}\mathscr{A}\}.$$

It is obvious that $\mathscr{U} \subseteq \mathscr{U}_\phi^r$. Conversely, let $T \in \mathscr{U}_\phi^r$. For any $x \in \mathscr{H}$, denote by E the orthogonal projection onto $[\mathscr{A}x]$. Then $E \in \operatorname{Lat}\mathscr{A}$, $x \in E$ and

$$Tx \in TE \subseteq \phi(E) = [\mathscr{U}E] = [\mathscr{U}[\mathscr{A}x]] = [\mathscr{U}x]$$

since $\mathscr{U}$ is a right $\mathscr{A}$-submodule. From the reflexivity of $\mathscr{U}$, it follows that $T \in \mathscr{U}$. Accordingly, $\mathscr{U}_\phi^r \subseteq \mathscr{U}$ and $\mathscr{U} = \mathscr{U}_\phi^r$.

(2) *Sufficiency.* Suppose that there exists $\psi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ such that

$$\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : T\psi(E) \subseteq E, \forall E \in \operatorname{Lat}\mathscr{A}\}.$$

Define $\phi : \operatorname{Lat}\mathscr{A}^* = (\operatorname{Lat}\mathscr{A})^\perp \to \mathscr{P}$ by

$$\phi(E^\perp) = I - \psi(E), \quad \forall E^\perp \in \operatorname{Lat}\mathscr{A}^* = (\operatorname{Lat}\mathscr{A})^\perp.$$

Certainly $\phi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}^*, \mathscr{P})$. Thus

$$\begin{aligned}\mathscr{U}^* &= \{T^* \in \mathscr{B}(\mathscr{H}) : T\psi(E) \subseteq E, \forall E \in \operatorname{Lat}\mathscr{A}\}\\ &= \{T^* \in \mathscr{B}(\mathscr{H}) : T^*E^\perp \subseteq \psi(E)^\perp = \phi(E^\perp), \forall E \in \operatorname{Lat}\mathscr{A}\}\\ &= \{S \in \mathscr{B}(\mathscr{H}) : SE^\perp \subseteq \phi(E^\perp), \forall E^\perp \in \operatorname{Lat}\mathscr{A}^* = (\operatorname{Lat}\mathscr{A})^\perp\}.\end{aligned}$$

It follows from (1) that $\mathscr{U}^*$ is a reflexive right $\mathscr{A}^*$-submodule, and $\mathscr{U}$ is a $*$-reflexive left $\mathscr{A}$-submodule.

Necessity. Suppose that $\mathscr{U}$ is a $*$-reflexive left $\mathscr{A}$-submodule. Thus $\mathscr{U}^*$ is a reflexive right $\mathscr{A}^*$-submodule, it follows from (1) that there exists $\phi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}^*, \mathscr{P})$ such that

$$\mathscr{U}^* = \{T \in \mathscr{B}(\mathscr{H}) : TE^\perp \subseteq \phi(E^\perp),\ \forall E^\perp \in \operatorname{Lat}\mathscr{A}^*\}.$$

Define $\psi : \operatorname{Lat}\mathscr{A} \to \mathscr{P}$ by $\psi(E) = I - \phi(E^\perp)$. Clearly $\psi \in \operatorname{Hom}(\operatorname{Lat}\mathscr{A}, \mathscr{P})$ and

$$\begin{aligned}\mathscr{U} &= \{T^* \in \mathscr{B}(\mathscr{H}) : T^*\phi(E^\perp)^\perp \subseteq E,\ \forall E \in \operatorname{Lat}\mathscr{A}\}\\ &= \{S \in \mathscr{B}(\mathscr{H}) : S\psi(E) \subseteq E,\ \forall E \in \operatorname{Lat}\mathscr{A}\}.\end{aligned}$$

□

From the proof of Theorem 2.1, we know that if $\mathscr{U}$ is a reflexive right $\mathscr{A}$-submodule then $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \tau_r(E), \forall E \in \operatorname{Lat}\mathscr{A}\}$, where $\tau_r(E) = [\mathscr{U}E]$; if $\mathscr{U}$ is a $*$-reflexive left $\mathscr{A}$-submodule then $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : T\tau_l(E) \subseteq E, \forall E \in \operatorname{Lat}\mathscr{A}\}$, where $\tau_l(E) = I - [\mathscr{U}^*E^\perp]$.

COROLLARY 2.2.

If $\mathscr{A}$ is a unital σ-weakly closed algebra which is σ-weakly generated by rank-one operators in $\mathscr{A}$, then every σ-weakly closed right or left $\mathscr{A}$-submodule has the form given in Theorem 2.1(1) *or* (2), *respectively.*

Proof. By virtue of ([5], Theorem 2.2), every σ-weakly closed right or left $\mathscr{A}$-submodule is reflexive. So the result is true for σ-weakly closed right $\mathscr{A}$-submodule by Theorem 2.1(1). Now for any σ-weakly closed left $\mathscr{A}$-submodule $\mathscr{U}$, since the adjoint operation is continuous in the σ-weak topology, $\mathscr{U}^*$ is a σ-weakly closed right $\mathscr{A}^*$-submodule and $\mathscr{A}^*$ is σ-weakly generated by rank-one operators in $\mathscr{A}^*$. Therefore it follows from ([5], Theorem 2.2) that $\mathscr{U}^*$ is reflexive and $\mathscr{U}$ is $*$-reflexive. Thus $\mathscr{U}$ has the form in Theorem 2.1(2). □

COROLLARY 2.3.

Suppose that $\mathscr{L}$ is a commutative and completely distributive subspace lattice, or specially, a nest. Then every σ-weakly closed right or left Alg $\mathscr{L}$*-submodule is of the form given in Theorem* 2.1(1) *or* (2), *respectively.*

Proof. This follows from Corollary 2.2 and ([7], Theorem 3). □

COROLLARY 2.4.

Suppose that $\mathscr{A}$ is a unital algebra in $\mathscr{B}(\mathscr{H})$.

(1) *Let $\mathscr{U}$ be as in* (1) *of Theorem* 2.1. *Then $\mathscr{U}$ is a right ideal if and only if $\tau_r(E) \leq E$ for every $E \in \mathrm{Lat}\,\mathscr{A}$, where $\tau_r(E) = [\mathscr{U}E]$;*
(2) *Let $\mathscr{U}$ be as in* (2) *of Theorem* 2.1. *Then $\mathscr{U}$ is a left ideal if and only if $\tau_l(E) \geq E$ for any $E \in \mathrm{Lat}\,\mathscr{A}$, where $\tau_l(E) = I - [\mathscr{U}^*E^{\perp}]$.*

Proof.

(1) Obvious.
(2) Let $\mathscr{U}$ be a left ideal of $\mathscr{A}$. Thus $\mathscr{U}^*$ is a right ideal of $\mathscr{A}^*$, it follows from (1) that $[\mathscr{U}^*E^{\perp}] \leq E^{\perp}$ for any $E^{\perp} \in \mathrm{Lat}\,\mathscr{A}^* = (\mathrm{Lat}\,\mathscr{A})^{\perp}$. This deduces that $\tau_l(E) = I - [\mathscr{U}^*E^{\perp}] \geq E$ for any $E \in \mathrm{Lat}\,\mathscr{A}$. The converse implication can be proved similarly. □

PROPOSITION 2.5.

Suppose that $\mathscr{A}$ is a unital algebra in $\mathscr{B}(\mathscr{H})$.

(1) *Let $\mathscr{U}$ be a reflexive right $\mathscr{A}$-submodule. Then $P \in \mathrm{Lat}\,\mathscr{U}$ if and only if there exists $E \in \mathrm{Lat}\,\mathscr{A}$ such that $\tau_r(E) \leq P \leq E$;*
(2) *Let $\mathscr{U}$ be a $*$-reflexive left $\mathscr{A}$-submodule. Then $P \in \mathrm{Lat}\,\mathscr{U}$ if and only if there exists $E \in \mathrm{Lat}\,\mathscr{A}$ such that $E \leq P \leq \tau_l(E)$.*

Proof.

(1) From the proof of Theorem 2.1, $\mathscr{U} = \{T \in \mathscr{B}(\mathscr{H}) : TE \subseteq \tau_r(E), E \in \mathrm{Lat}\,\mathscr{A}\}$. If $\tau_r(E) \leq P \leq E$ for some $E \in \mathrm{Lat}\,\mathscr{A}$ and $T \in \mathscr{U}$, then

$$TP = TEP = \tau_r(E)TEP = P\tau_r(E)TEP = PTP.$$

So $P \in \mathrm{Lat}\,\mathscr{U}$.

Conversely, if $P \in \mathrm{Lat}\,\mathscr{U}$, let $E = [\mathscr{A}P]$. Then $E \in \mathrm{Lat}\,\mathscr{A}$, $E \geq P$ and

$$\tau_r(E) = [\mathscr{U}E] = [\mathscr{U}[\mathscr{A}P]] \subseteq [\mathscr{U}P] \subseteq P$$

since $\mathscr{U}$ is a right $\mathscr{A}$-module. Thus $\tau_r(E) \leq P \leq E$.
(2) Follows from (1) and a simple calculation. □

For non-zero vectors $x, y \in \mathscr{H}$, the rank-one operator $x \otimes y$ is defined by the equation

$$(x \otimes y)z = \langle z, y\rangle x, \quad \forall z \in \mathscr{H}.$$

Lemma 2.6. *Suppose that $\mathscr{A}$ is a unital algebra in $\mathscr{B}(\mathscr{H})$.*

(1) *Let $\mathscr{U}_\phi^r$ be the reflexive right $\mathscr{A}$-submodule determined by ϕ in* $\mathrm{Hom}(\mathrm{Lat}\,\mathscr{A}, \mathscr{P})$. *Then a rank-one operator $x \otimes y \in \mathscr{U}_\phi^r$ if and only if for some $E \in \mathrm{Lat}\,\mathscr{A}$, $x \in E$ and $y \in \phi_\sim(E)^\perp$, where $\phi_\sim(E) = \vee\{F \in \mathrm{Lat}\,\mathscr{A} : \phi(F) \ngeq E\}$.*

(2) *Let $\mathscr{U}_\phi^l$ be the $*$-reflexive left $\mathscr{A}$-submodule determined by ϕ in* $\mathrm{Hom}(\mathrm{Lat}\,\mathscr{A}, \mathscr{P})$. *Then a rank-one operator $x \otimes y \in \mathscr{U}_\phi^l$ if and only if for some $E \in \mathrm{Lat}\,\mathscr{A}$, $x \in \wedge\{F \in \mathrm{Lat}\,\mathscr{A} : \phi(F) \nleq E\}$ and $y \in E^\perp$.*

Proof.

(1) Suppose that there exists $E \in \mathrm{Lat}\,\mathscr{A}$ such that $x \in E$ and $y \in \phi_\sim(E)^\perp$. For any $F \in \mathrm{Lat}\,\mathscr{A}$, if $\phi(F) \geq E$, then

$$(x \otimes y)F = E(x \otimes y)\phi_\sim(E)^\perp F \subseteq E \subseteq \phi(F);$$

if $\phi(F) \ngeq E$, it follows from the definition of $\phi_\sim(E)$ that $F \leq \phi_\sim(E)$. Thus

$$(x \otimes y)F = E(x \otimes y)\phi_\sim(E)^\perp F = 0 \subseteq \phi(F).$$

Accordingly, $x \otimes y \in \mathscr{U}_\phi^r$.

Conversely, if $x \otimes y \in \mathscr{U}_\phi^r$. Let

$$E = \wedge\{F \in \mathrm{Lat}\,\mathscr{A} : Fx = x\}.$$

Naturally, $E \in \mathrm{Lat}\,\mathscr{A}$ and $x \in E$. For any $F \in \mathrm{Lat}\,\mathscr{A}$ and $\phi(F) \ngeq E$, it follows from the definition of E that $\phi(F)x \neq x$. Since $x \otimes y \in \mathscr{U}_\phi^r$, we have

$$(x \otimes y)Fy = \phi(F)(x \otimes y)Fy$$

and

$$\|Fy\|^2 x = \|Fy\|^2 \phi(F)x.$$

So $Fy = 0$. From the definition of $\phi_\sim(E)$, it follows that $\phi_\sim(E)y = 0$ and $y \in \phi_\sim(E)^\perp$.

(2) By hypothesis, $\mathscr{U}_\phi^l = \{T \in \mathscr{B}(\mathscr{H}) : T\phi(E) \subseteq E, \forall E \in \mathrm{Lat}\,\mathscr{A}\}$. Define $\psi : \mathrm{Lat}\,\mathscr{A}^* \to \mathscr{P}$ by $\psi(E^\perp) = I - \phi(E)$. Thus $\psi \in \mathrm{Hom}\,(\mathrm{Lat}\,\mathscr{A}^*, \mathscr{P})$ and

$$\begin{aligned}
(\mathscr{U}_\phi^l)^* &= \{T^* \in \mathscr{B}(\mathscr{H}) : T\phi(E) \subseteq E, \forall E \in \mathrm{Lat}\,\mathscr{A}\} \\
&= \{T^* \in \mathscr{B}(\mathscr{H}) : T^*E^\perp \subseteq \phi(E)^\perp, \forall E \in \mathrm{Lat}\,\mathscr{A}\} \\
&= \{S \in \mathscr{B}(\mathscr{H}) : SE^\perp \subseteq \psi(E^\perp), \forall E^\perp \in \mathrm{Lat}\,\mathscr{A}^*\}.
\end{aligned}$$

$(\mathscr{U}_\phi^l)^*$ is a reflexive right $\mathscr{A}^*$-submodule determined by ψ. From (1), it follows that $y \otimes x \in (\mathscr{U}_\phi^l)^*$ if and only if there exists $E^\perp \in \mathrm{Lat}\,\mathscr{A}^*$ such that $y \in E^\perp$ and $x \in \psi_\sim(E^\perp)^\perp$. Now we compute $\psi_\sim(E^\perp)^\perp$. It follows from the definition of $\psi_\sim$ that

$$\begin{aligned}
\psi_\sim(E^\perp)^\perp &= (\vee\{F^\perp \in \mathrm{Lat}\,\mathscr{A}^* : \psi(F^\perp) \ngeq E^\perp\})^\perp \\
&= \wedge\{F \in \mathrm{Lat}\,\mathscr{A} : \phi(F)^\perp \ngeq E^\perp\} \\
&= \wedge\{F \in \mathrm{Lat}\,\mathscr{A} : \phi(F) \nleq E\}.
\end{aligned}$$

□

3. Rank-one operators

In this section, we only consider the reflexive right $\mathscr{A}$-submodule, and omit the superscript and subscript r in the corresponding notation. The corresponding results for $*$-reflexive left $\mathscr{A}$-submodule hold naturally, we leave the details for the interested readers.

Theorem 3.1. *Suppose that $\mathscr{U}_\phi$ is a reflexive right $\mathscr{A}$-submodule determined by ϕ in $\operatorname{Hom}(\operatorname{Lat}\mathscr{A},\mathscr{P})$ and $\mathscr{R}_\phi$ the rank-one submodule generated by rank-one operators in $\mathscr{U}_\phi$. Then $K\in\operatorname{Lat}\mathscr{R}_\phi$ if and only if there exists $E\in\operatorname{Lat}\mathscr{A}$ such that $E\le K\le\phi_*(E)$, where $\phi_*(E)=\wedge\{\phi_\sim(F):F\in\operatorname{Lat}\mathscr{A},F\not\le E\}$.*

Proof. Suppose that $K\in\operatorname{Lat}\mathscr{R}_\phi$. Let $E=\vee\{F\in\operatorname{Lat}\mathscr{A}:F\le K\}$. Then $E\in\operatorname{Lat}\mathscr{A}$ and $E\le K$. Let $F\in\operatorname{Lat}\mathscr{A}$ with $F\not\le E$. We will show that $K\le\phi_\sim(F)$. Let y be any element of K. Now $F\not\le K$. So we can choose a vector $e\in F$ and $e\notin K$. Since $K\in\operatorname{Lat}\mathscr{R}_\phi$, for every vector $f\in\phi_\sim(F)^\perp$, we have $(e\otimes f)y=(y,f)e\in K$. But since $e\notin K$ it follows that $(y,f)=0$ and $y\in\phi_\sim(F)$. Thus $K\le\phi_\sim(F)$ and so $K\le\phi_*(E)$.

Now suppose that there is a subspace $E\in\operatorname{Lat}\mathscr{A}$ with $E\le K\le\phi_*(E)$. Let $e\otimes f\in\mathscr{R}_\phi$. By Lemma 2.6(1) there is an element $F\in\operatorname{Lat}\mathscr{A}$ such that $e\in F$ and $f\in\phi_\sim(F)^\perp$. If $F\le E$ then $(e\otimes f)K\subseteq F\subseteq E\subseteq K$. If $F\not\le E$ then $K\le\phi_*(E)\le\phi_\sim(F)$ and $(e\otimes f)K=(0)\subseteq K$. In either case K is invariant under $e\otimes f$ and $K\in\operatorname{Lat}\mathscr{R}_\phi$. □

Suppose that $\mathscr{U}$ is a reflexive right $\mathscr{A}$-module and $\mathscr{R}$ is the rank-one submodule of $\mathscr{U}$, it follows from Theorems 2.1 and 3.1 that $K\in\operatorname{Lat}\mathscr{R}$ if and only if for some $E\in\operatorname{Lat}\mathscr{A}$, $E\le K\le\tau_*(E)$, where $\tau(E)=[\mathscr{U}E]$.

COROLLARY 3.2.

If $\mathscr{L}$ is a subspace lattice, $\mathscr{R}$ is the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$. Then $K\in\operatorname{Lat}\mathscr{R}$ if and only if for some $E\in\operatorname{Lat}\operatorname{Alg}\mathscr{L}$, $E\le K\le E_*$, where $E_*=\wedge\{F_-:F\in\operatorname{Lat}\operatorname{Alg}\mathscr{L},F\not\le E\}$ and $F_-=\vee\{G\in\operatorname{Lat}\operatorname{Alg}\mathscr{L}:G\not\ge F\}$.

Proof. In this case, $\tau(E)=[(\operatorname{Alg}\mathscr{L})E]=E$ and $\tau_\sim(E)=\vee\{G\in\operatorname{Lat}\operatorname{Alg}\mathscr{L}:\tau(G)\not\ge E\}=\vee\{G\in\operatorname{Lat}\operatorname{Alg}\mathscr{L}:G\not\ge E\}=E_-$ and $\tau_*(E)=\wedge\{\tau_\sim(F):F\in\operatorname{Lat}\operatorname{Alg}\mathscr{L},F\not\le E\}=\wedge\{F_-:F\in\operatorname{Lat}\operatorname{Alg}\mathscr{L},F\not\le E\}=E_*$. The corollary follows from Theorem 3.1. □

If $\mathscr{L}$ is a subspace lattice and $E\in\mathscr{L}$, we define $E_-^{\mathscr{L}}=\vee\{F\in\mathscr{L}:F\not\ge E\}$ and $E_*^{\mathscr{L}}=\wedge\{F_-^{\mathscr{L}}:F\in\mathscr{L},F\not\le E\}$. The following proposition is due to Longstaff. It gives a similar characterization of $\operatorname{Lat}\mathscr{R}$ by means of the elements in $\mathscr{L}$.

The next two theorems comprise some of the main results of this paper.

Theorem 3.3. *Suppose that $\mathscr{L}$ is a subspace lattice and $\mathscr{R}$ is the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$. The following statements are equivalent.*

(1) $\operatorname{Alg}\mathscr{L}=\operatorname{Alg}\operatorname{Lat}\mathscr{R}$;
(2) $\operatorname{Lat}\operatorname{Alg}\mathscr{L}=\operatorname{Lat}\mathscr{R}$;
(3) $[E,E_*]\subseteq\operatorname{Lat}\operatorname{Alg}\mathscr{L}$ *for any* $E\in\operatorname{Lat}\operatorname{Alg}\mathscr{L}$, *where* $[E,E_*]=\{K\in\mathscr{P}:E\le K\le E_*\}$.

Proof. It is clear that (1) is equivalent to (2), we only need to show that (2) is equivalent to (3).

$(2) \Rightarrow (3)$. By definition, $E_* = \wedge\{F_- : F \in \operatorname{Lat}\operatorname{Alg}\mathscr{L}, F \not\leq E\}$. It follows from the definition of F_- that $E \leq F_-$ for any $F \not\leq E$. Thus $E \leq E_*$ for any $E \in \operatorname{Lat}\operatorname{Alg}\mathscr{L}$, so the symbol $[E, E_*]$ is meaningful. For any $E \in \operatorname{Lat}\operatorname{Alg}\mathscr{L}$ and $K \in [E, E_*]$, it follows from Corollary 3.2 that $K \in \operatorname{Lat}\mathscr{R} = \operatorname{Lat}\operatorname{Alg}\mathscr{L}$. Hence $[E, E_*] \subseteq \operatorname{Lat}\operatorname{Alg}\mathscr{L}$ for any $E \in \operatorname{Lat}\operatorname{Alg}\mathscr{L}$.
$(3) \Rightarrow (2)$. Since $\mathscr{R} \subseteq \operatorname{Alg}\mathscr{L}$, $\operatorname{Lat}\operatorname{Alg}\mathscr{L} \subseteq \operatorname{Lat}\mathscr{R}$. For any $K \in \operatorname{Lat}\mathscr{R}$, it follows from Corollary 3.2 that there is an element $E \in \operatorname{Lat}\operatorname{Alg}\mathscr{L}$ such that $E \leq K \leq E_*$. So $K \in [E, E_*] \subseteq \operatorname{Lat}\operatorname{Alg}\mathscr{L}$. Thus $\operatorname{Lat}\mathscr{R} \subseteq \operatorname{Lat}\operatorname{Alg}\mathscr{L}$ and $\operatorname{Lat}\mathscr{R} = \operatorname{Lat}\operatorname{Alg}\mathscr{L}$. □

PROPOSITION 3.4.([8], Proposition 3.2)

Suppose that $\mathscr{R}$ is the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$. *Then the subspace K belongs to* $\operatorname{Lat}\mathscr{R}$ *if and only if there is a subspace E of $\mathscr{L}$ such that $E \leq K \leq E_*^{\mathscr{L}}$.*

COROLLARY 3.5.

Suppose that $\mathscr{L}$ is a subspace lattice and $\mathscr{R}$ is the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$. *Then* $\mathscr{L} = \operatorname{Lat}\mathscr{R}$ *if and only if $[E, E_*^{\mathscr{L}}] \subseteq \mathscr{L}$ for any $E \in \mathscr{L}$.*

Proof. Suppose that $\mathscr{L} = \operatorname{Lat}\mathscr{R}$. For any $E \in \mathscr{L}$, we can show $E \leq E_*^{\mathscr{L}}$ similarly as in Theorem 3.3. For $K \in [E, E_*^{\mathscr{L}}]$, it follows from Proposition 3.4 that $K \in \operatorname{Lat}\mathscr{R} = \mathscr{L}$. So $[E, E_*^{\mathscr{L}}] \subseteq \mathscr{L}$ for any $E \in \mathscr{L}$.

Conversely, if $[E, E_*^{\mathscr{L}}] \subseteq \mathscr{L}$ for any $E \in \mathscr{L}$. For any $K \in \operatorname{Lat}\mathscr{R}$, it follows from Proposition 3.4 that there is an element $E \in \mathscr{L}$ such that $E \leq K \leq E_*^{\mathscr{L}}$. So $K \in [E, E_*^{\mathscr{L}}] \subseteq \mathscr{L}$. Thus $\operatorname{Lat}\mathscr{R} \subseteq \mathscr{L}$. Combining with the fact that $\mathscr{L} \subseteq \operatorname{Lat}\operatorname{Alg}\mathscr{L} \subseteq \operatorname{Lat}\mathscr{R}$, we obtain $\mathscr{L} = \operatorname{Lat}\mathscr{R}$. □

COROLLARY 3.6.

Suppose that $\mathscr{L}$ is a subspace lattice. If $[E, E_*] \subseteq \mathscr{L}$ for any $E \in \mathscr{L}$, then $\mathscr{L}$ is a reflexive subspace lattice.

Proof. From Corollary 3.5, it follows that $\mathscr{L} = \operatorname{Lat}\mathscr{R}$. Since $\mathscr{L} \subseteq \operatorname{Lat}\operatorname{Alg}\mathscr{L} \subseteq \operatorname{Lat}\mathscr{R}$, so $\mathscr{L} = \operatorname{Lat}\operatorname{Alg}\mathscr{L}$ and $\mathscr{L}$ is reflexive. □

Proposition 3.4, and its Corollary 3.5, answer the question of which subspace lattices $\mathscr{L}$ are determined by the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$ in the sense that $\mathscr{L} = \operatorname{Lat}\mathscr{R}$. This proposition was used as the basis of an abstract, lattice-theoretic, way of constructing reflexive lattices in [10]. Theorem 3.3 gives a sufficient and necessary condition for which reflexive algebra $\operatorname{Alg}\mathscr{L}$ is determined by the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$ in the sense that $\operatorname{Alg}\mathscr{L} = \operatorname{Alg}\operatorname{Lat}\mathscr{R}$. In the following, we will consider another sense that $\operatorname{Alg}\mathscr{L}$ is determined by the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$.

Theorem 3.7. *Suppose that $\mathscr{A}$ is a unital algebra in $\mathscr{B}(\mathscr{H})$ and $\mathscr{U}$ is a reflexive right $\mathscr{A}$-submodule, and that $\mathscr{R}$ the rank-one submodule of $\mathscr{U}$. Then* $\operatorname{Ref}\mathscr{R} = \mathscr{U}$ *if and only if $\tau = (\tau_\sim)_\sim$, where $\tau(E) = [\mathscr{U}E]$ for any $E \in \operatorname{Lat}\mathscr{A}$.*

Proof. Necessity. Recall that the reflexive hull $\operatorname{Ref}\mathscr{R} = \{T \in \mathscr{B}(\mathscr{H}) : Tx \in [\mathscr{R}x], \forall x \in \mathscr{H}\}$ for any $E \in \operatorname{Lat}\mathscr{A}$ and $e \otimes f \in \mathscr{U}$. We first show

$$(e \otimes f)E \subseteq (\tau_\sim)_\sim(E) = \vee\{F \in \operatorname{Lat}\mathscr{A} : \tau_\sim(F) \not\geq E\}.$$

By virtue of Lemma 2.6(1), there is an element $L \in \operatorname{Lat}\mathscr{A}$ such that $e \in L$ and $f \in \tau_\sim(L)^\perp$. If $\tau_\sim(L) \geq E$ then

$$(e \otimes f)E = L(e \otimes f)\tau_\sim(L)^\perp E = (0) \subseteq (\tau_\sim)_\sim(E);$$

if $\tau_{\sim}(L) \not\geq E$ then $L \leq (\tau_{\sim})_{\sim}(E)$. Thus

$$(e \otimes f)E = L(e \otimes f)\tau_{\sim}(L)^{\perp}E \subseteq L \subseteq (\tau_{\sim})_{\sim}(E).$$

So each rank-one operator of $\mathscr{U}$ maps E into $(\tau_{\sim})_{\sim}(E)$ for any $E \in \operatorname{Lat}\mathscr{A}$. For any $A \in \mathscr{U}$ and $x \in E (\in \operatorname{Lat}\mathscr{A})$, since $\mathscr{U} = \operatorname{Ref}\mathscr{R} = \{T \in \mathscr{B}(\mathscr{H}) : Tx \in [\mathscr{R}x], \forall x \in \mathscr{H}\}$, so $Ax \in [\mathscr{R}x] \subseteq [\mathscr{R}E] \subseteq (\tau_{\sim})_{\sim}(E)$ and $AE \subseteq (\tau_{\sim})_{\sim}(E)$. Accordingly, $\tau(E) = [\mathscr{U}E] \subseteq (\tau_{\sim})_{\sim}(E)$ and $\tau \leq (\tau_{\sim})_{\sim}$. For any $E \in \operatorname{Lat}\mathscr{A}$, it follows from the definitions that

$$(\tau_{\sim})_{\sim}(E) = \vee\{G \in \operatorname{Lat}\mathscr{A} : \tau_{\sim}(G) \not\geq E\} \tag{1}$$

and

$$\tau_{\sim}(G) = \vee\{F \in \operatorname{Lat}\mathscr{A} : \tau(F) \not\geq G\}. \tag{2}$$

For $G \in \operatorname{Lat}\mathscr{A}$ and $\tau_{\sim}(G) \not\geq E$, if $\tau(E) \not\geq G$ then it follows from (2) that $\tau_{\sim}(G) \geq E$. This contradiction shows that $\tau(E) \geq G$. Thus eq. (1) tells us that $\tau \geq (\tau_{\sim})_{\sim}$. Hence $\tau = (\tau_{\sim})_{\sim}$.

Sufficiency. Suppose that $\tau = (\tau_{\sim})_{\sim}$. It is clear that $\operatorname{Ref}\mathscr{R} \subseteq \mathscr{U}$, it suffices to show that $\operatorname{Ref}\mathscr{R} \supseteq \mathscr{U}$. Suppose that $A \in \mathscr{U}$ and $x \in \mathscr{H}$. From the definition $\operatorname{Ref}\mathscr{R}$, we only need to prove that $Ax \in [\mathscr{R}x]$.

Define E by $E = \wedge\{F \in \operatorname{Lat}\mathscr{A} : x \in F\}$. Observe that the intersection is over a non-empty family of subspaces of $\operatorname{Lat}\mathscr{A}$ since $x \in \mathscr{H}$. Clearly $x \in E$ and $E \in \operatorname{Lat}\mathscr{A}$. By the hypothesis,

$$\tau(E) = [\mathscr{U}E] = \vee\{G \in \operatorname{Lat}\mathscr{A} : \tau_{\sim}(G) \not\geq E\}$$

and hence the set of all $G \in \operatorname{Lat}\mathscr{A}$ with $\tau_{\sim}(G) \not\geq E$ has a dense linear span in $[\mathscr{U}E]$. Therefore for any $\varepsilon > 0$, there is a finite set $G_i (1 \leq i \leq n)$ of subspaces of $\operatorname{Lat}\mathscr{A}$ with $\tau_{\sim}(G_i) \not\geq E$ and a set of vectors $x_i \in G_i (1 \leq i \leq n)$ with the property that

$$\|Ax - (x_1 + \cdots + x_n)\| < \varepsilon.$$

The definition of E and the condition $\tau_{\sim}(G_i) \not\geq E (1 \leq i \leq n)$ implies that $x \notin \tau_{\sim}(G_i) (1 \leq i \leq n)$ and so there exists $y_i \in \tau_{\sim}(G_i)^{\perp}$ with

$$\langle x, y_i \rangle \neq 0, \qquad \forall 1 \leq i \leq n.$$

By suitably scaling y_i if needed we may assume that $\langle x, y_i \rangle = 1$ and so $(x_i \otimes y_i)x = x_i$ for $1 \leq i \leq n$. By Lemma 2.6(1), $x_i \otimes y_i \in \mathscr{R}$. Thus

$$\left\| Ax - \left(\sum_{i=1}^{n} x_i \otimes y_i \right) x \right\| = \|Ax - (x_1 + \cdots + x_n)\| < \varepsilon,$$

and this shows that $Ax \in [\mathscr{R}x]$ and $A \in \operatorname{Ref}\mathscr{R}$. Hence $\operatorname{Ref}\mathscr{R} = \mathscr{U}$. □

COROLLARY 3.8.

Suppose that $\mathscr{L}$ is a subspace lattice and $\mathscr{R}$ is the rank-one subalgebra of $\operatorname{Alg}\mathscr{L}$. *Then $\mathscr{L}$ is completely distributive if and only if* $\operatorname{Ref}\mathscr{R} = \operatorname{Alg}\mathscr{L}$.

Proof. In this case, $\tau(E) = [(\mathrm{Alg}\,\mathscr{L})E] = E$, $\tau_{\sim}(E) = E_{-}$ and $(\tau_{\sim})_{\sim}(E) = \vee\{F \in \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L} : \tau_{\sim}(F) \not\geq E\} = \vee\{F \in \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L} : F_{-} \not\geq E\} = E_{\sharp}$. Now suppose that $\mathrm{Ref}\,\mathscr{R} = \mathrm{Alg}\,\mathscr{L}$. So from Theorem 3.7, it follows that $E = E_{\sharp}$ for any $E \in \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$. Theorem 5.2 of ref. [8] shows that $\mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$ is completely distributive. $\mathscr{L} \subseteq \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$ implies that $\mathscr{L}$ is also completely distributive.

Conversely, suppose that $\mathscr{L}$ is completely distributive. From ([8], Theorem 6.1), it follows that $\mathscr{L} = \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$. So $E = E_{\sharp}$ for any $E \in \mathscr{L} = \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$. It follows from Theorem 3.7 that $\mathrm{Ref}\,\mathscr{R} = \mathrm{Alg}\,\mathscr{L}$. □

Corollary 3.8 was first proved by Lambrou ([6], Theorem 3.1). From the above proof, we can easily obtain that $\mathrm{Ref}\,\mathscr{R} = \mathrm{Alg}\,\mathscr{L}$ is equivalent to the complete distributivity of $\mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$. Thus it follows from ([8], Theorem 5.2) that $\mathrm{Ref}\,\mathscr{R} = \mathrm{Alg}\,\mathscr{L}$ if and only if $E = E_{*}$ for any $E \in \mathrm{Lat}\,\mathrm{Alg}\,\mathscr{L}$. Comparing with Theorem 3.3, shows the differences between $\mathrm{Alg}\,\mathrm{Lat}\,\mathscr{R}$ and $\mathrm{Ref}\,\mathscr{R}$.

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